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An inverse numerical method by reproducing kernel Hilbert spaces and its application to Cauchy problem for an elliptic equation. (再生核ヒルベルト空間による逆問題数値計算法と楕円型方程式のコーシー問題への応用)

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Abstract

We propose a discretized Tikhonov regularization for a Cauchy problem for an elliptic equation by a reproducing kernel Hilbert space. We prove the convergence of discretized regularized solutions to an exact solution. Our numerical results demonstrate that our method can stably reconstruct solutions to the Cauchy problems even in severe cases of geometric configurations.

1 Discretized Tikhonov regularization

Many inverse problems can be reduced to a linear ill-posed operator equation:

$$Kf = g, \tag{1}$$

by choosing suitably Hilbert spaces V and W and a linear compact operator $K\colon V \to W$. Henceforth $(\cdot,\cdot)_V$ means the inner product in V, and by $\|\cdot\|_V$ we denote the norm in V if we need to specify the space V.

We aim at the reconstruction of f_0 satisfying $Kf_0 = g_0$ by means of noisy data g_δ satisfying $\|g_0 - g_\delta\|_W \le \delta$, where $\delta > 0$ is a noise level. We assume that the value of δ is known *a priori.*

In order to stably reconstruct f_0 from some noisy data g_δ, we consider the Tikhonov regularization [13]. Let V_m be a finite dimensional linear subspace. Let $\{f_j^m\}_{1\le j\le m}$ be a linearly independent set of V_m. We denote P_m to be the orthogonal projection of V onto V_m. Moreover, we define the function spaces $W_m \subset W$ by $W_m := \mathrm{span}\{K(f_j^m) \mid f_j^m \in V_m\ j = 1,\dots,m\}$. For

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any given g_0, we expand g_0 in the finite subspace W_m. This is done by considering the minimization problems $\min_{g\in W_m} \|g - g_\delta\|_W = \min_{f\in V_m} \|K(f) - g_\delta\|_W$. Once the expanded coefficients of $f_{\min} := \arg\min_{f\in V_m} \|K(f) - g_\delta\|_W$ are obtained, we can regard $f_{\min}$ as an approximation to f_0. However, due to the ill-posedness of the compact operator K, the function $f_{\min}$ needs not approximate the solution f_0 reasonably even when $g_\delta = g_0$. In order to overcome this difficulty, we introduce the regularization term with the norm of V. Thus, we arrive at a discretized Tikhonov regularization on the finite dimensional space V_m:

$$\min_{f\in V_m} \|Kf - g_\delta\|_W^2 + \alpha\|f\|_V^2, \tag{2}$$

where $\alpha > 0$ is called the regularization parameter. The formulation (2) corresponds to a Ritz approach in [4] where $V_m \subset V_{m+1}$ is assumed.

We know that there exists a unique minimizer $f_{\alpha,m,\delta}$ of (2) for any $\alpha > 0$, $\delta > 0$ and $m \in \mathbb{N}$. Moreover, the minimizer is given by

$$f_{\alpha,m,\delta} = (K_m^* K_m + \alpha I)^{-1} K_m^* g_\delta,$$

where $K_m = KP_m$. We denote the minimizer when $\delta = 0$ by $f_{\alpha,m}$. With some *a priori* choices of α and m for given $\delta > 0$, we can prove the convergence of the Tikhonov regularized solutions.

We can now prove the convergence of the minimizer (2) to the solution $K^\dagger g_0$, where $K^\dagger g_0$ is the unique minimum least-squares solution for $\min_{f\in V} \|Kf - g_0\|$. Let $\gamma_m = \|K(I - P_m)\|$.

Proposition 1 ([12]). *Suppose that* $\lim_{m\to\infty} \gamma_m = 0$.

1. *Let* $\lim_{m\to\infty} \alpha_m = 0$. *If* $\gamma_m = O(\sqrt{\alpha_m})$, *then* $\lim_{m\to\infty} f_{\alpha_m,m} = K^\dagger g_0$ *in* V.
2. *Suppose that* $\lim_{m\to\infty} \|(I - P_m)f\| = 0$ *for all* $f \in V$. *Let* $\lim_{\delta\to 0} m(\delta) = \infty$ *and* $\lim_{\delta\to 0} \alpha(\delta) = 0$. *If* $\gamma_m = O(\sqrt{\alpha})$, $\delta = O(\sqrt{\alpha})$, *then* $\lim_{\delta\to 0} f_{\alpha(\delta),m(\delta),\delta} = K^\dagger g_0$ *weakly in* V.

2 Reproducing kernel Hilbert spaces

In this section, we introduce a reproducing kernel Hilbert space. One can refer to [1, 11, 14] for detailed treatises.

Let E be an arbitrary non-empty subset of $\mathbb{R}^d$. We call a symmetric function $\Phi\colon E \times E \to \mathbb{R}$ a *kernel*. A kernel Φ is said to be *positive definite* (respectively, *positive semi-definite)*, if for all $N \in \mathbb{N}$ and all sets of pairwise distinct points $X = \{x_1, \ldots, x_N\} \subset E$, the matrix $[\Phi(x_i, x_j)]_{i,j}$ is positive definite (respectively, positive semi-definite).

Definition 2. Let $\mathcal{H}$ be a real Hilbert space with the inner product $(\cdot,\cdot)_{\mathcal{H}}$ whose elements are some real-valued functions defined in E. A function $\Phi\colon E\times E\to\mathbb{R}$ is called a *reproducing kernel* for $\mathcal{H}$ if

1. $\Phi(\cdot,x)\in\mathcal{H}$ for all $x\in E$,

2. $f(x)=(f,\Phi(\cdot,x))_{\mathcal{H}}$. for all $f\in\mathcal{H}$ and all $x\in E$.

We define the norm by $\|f\|_{\mathcal{H}}=(f,f)^{\frac{1}{2}}_{\mathcal{H}}$.

A Hilbert space of functions that admits a reproducing kernel is called a *reproducing kernel Hilbert space* (in short, *RKHS*).

For a finite set of points $X:=\{x_1,\ldots,x_N\}$ and $f\in\mathcal{H}$, we define $s_{f,X}(x)$ by $s_{f,X}(x):=\sum_{k=1}^{N}\alpha_k\Phi(x,x_k)$, where the coefficients $\{\alpha_k\}_{k=1}^{N}$ are determined by the conditions $s_{f,X}(x_k)=f(x_k),\quad 1\le k\le N$. Since the matrix $[\Phi(x_i,x_j)]_{i,j}$ is positive definite, $\{\alpha_k\}_{k=1}^{N}$ are uniquely determined.

We define a subspace by $\mathcal{V}_X:=\operatorname{span}\{\Phi(\cdot,x)\mid x\in X\}\subset\mathcal{H}$, and an operator $P_X\colon\mathcal{H}\to\mathcal{V}_N\subset\mathcal{H}\ P_X(f)(x)=s_{f,X}(x)$.

Proposition 3 (see [14]). *P_X is an orthogonal projection of $\mathcal{H}$ onto the closed subspace $\mathcal{V}_X$.*

Define the fill distance h_X of X by $h_{X,E}=\sup_{x\in E}\min_{x_j\in X}|x-x_j|$. We choose some finite sets of points X_m, $m\in\mathbb{N}$ of E such that $h_{X_m,E}>h_{X_{m'},E}$ for all $m<m'\in\mathbb{N}$ and $\lim_{m\to\infty}h_{X_m,E}=0$. We set $V_m:=\mathcal{V}_{X_m}$ and $P_m:=P_{V_m}$. In general, we cannot guarantee that the union $\bigcup_{m=1}^{\infty}V_m$ is dense in $\mathcal{H}$ nor $\lim_{m\to\infty}\|f-P_m(f)\|_{\mathcal{H}}=0$. However, with a moderate assumption on the kernel Φ, we can prove these properties, which are crucial in our regularization method.

Lemma 4 ([12]). *If the reproducing kernel Φ is uniformly continuous on $E\times E$, then we have*

1. $\bigcup_{m=1}^{\infty}V_m$ *is dense in* $\mathcal{H}$.

2. $\lim_{m\to\infty}\|f-P_m(f)\|_{\mathcal{H}}=0$ *for all* $f\in\mathcal{H}$.

3 Discretized Tikhonov regularization by reproducing kernel Hilbert spaces

In this section, we apply the general results to the case when V is a RKHS.

Let E be a subset of $\mathbb{R}^d$. Let $(E,\mathcal{F},\mu)$ be a measure space on E. Let $\Phi\colon E\times E\to\mathbb{R}$ be a reproducing kernel. We assume that Φ is uniformly continuous on $E \times E$. We define a RKHS $\mathcal{H}$ on E generated by the kernel Φ. Let $K\colon \mathcal{H} \to W$ be a linear compact operator, where W is a Hilbert space. We consider the problem of finding the solution $f_0 \in \mathcal{H}$ in $Kf_0 = g_0$ by means of noisy data g_δ satisfying

$$\|g - g_\delta\|_W \le \delta.$$

We choose finite sets of points X_m, $m \in \mathbf{N}$ of E such that $\lim_{m\to\infty} h_{X_m,E} = 0$. We set a finite dimensional subspace $V_m := \mathcal{V}_{X_m}$ and the projection $P_m := P_{V_m}$. By Lemma 4, we have $\lim_{m\to\infty} \|(I - P_m)f\| = 0$ for all $f \in \mathcal{H}$. Set $\gamma_m = \|K(I - P_m)\|$. Henceforth we assume that $\lim_{m\to\infty} \gamma_m = 0$, which is satisfied by many reproducing kernels [14].

Let $f_{\alpha,m,\delta}$ be a unique solution of (2) when $V = \mathcal{H}$ and let $f_{\alpha,m}$ be a unique solution of (2) when the data $g_\delta = g_0$. From the results obtained above and the property of a RKHS, we have the following results.

Theorem 5 ([12]). *Under the above settings, we have the followings:*

1. *Let* $\lim_{m\to\infty} \alpha_m = 0$. *Suppose* $\sup_{x\in E} \Phi(x,x) < \infty$.

   If $\gamma_m = O(\sqrt{\alpha_m})$, *then* $\lim_{m\to\infty} \|f_{\alpha_m,m} - K^\dagger g_0\|_{L^\infty(E,\mu)} = 0$.

2. *Let* $\lim_{\delta\to 0} m(\delta) = \infty$ *and* $\lim_{\delta\to 0} \alpha(\delta) = 0$. *Suppose* $\int_E \Phi(x,x)d\mu(x) < \infty$.

   If $\gamma_m = O(\sqrt{\alpha})$, $\delta = O(\sqrt{\alpha})$, *then* $\lim_{\delta\to 0} \|f_{\alpha(\delta),m(\delta),\delta} - K^\dagger g_0\|_{L^2(E,\mu)} = 0$.

4 Tikhonov regularization by a reproducing kernel Hilbert space for the Cauchy problem for an elliptic equation

In this section, we consider a classical ill-posed problem, the Cauchy problem for an elliptic equation: Given h, g_1 and g_2, find u inside of Ω or $u|_{\partial\Omega\setminus\Gamma}$ where

$$\begin{cases} Au = h, & x \in \Omega, \\ u|_\Gamma = g_1, & \\ \partial_A u|_\Gamma = g_2, & \end{cases} \tag{3}$$

In (3), the domain $\Omega \subset \mathbb{R}^n$ is a bounded domain whose boundary $\partial\Omega$ is of C^2 class, Γ is a relatively open subset of $\partial\Omega$, and $Au(x) = \sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j u(x)) + c(x)u$, $x \in \Omega$, $\nu = \nu(x)$ is the unit outward normal vector to $\partial\Omega$ at x, $\partial_A u = \sum_{i,j=1}^n a_{ij}(x)(\partial_j u)\nu_i$. Moreover, we assume that $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $1 \le i, j \le n$, $c \in L^\infty(\Omega)$ and that there exists a constant $\gamma_0 > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \gamma_0 \sum_{j=1}^n \xi_j^2, \quad x \in \overline{\Omega},\ \xi_1, \ldots, \xi_n \in \mathbb{R}.$$

This problem appears in many applications for example in the cardiography, the nondestructive testing, etc. Stable and efficient numerical methods are of high importance. However, it is well-known that the Cauchy problem for an elliptic equation is ill-posed without any *a priori* bounds of u (e.g., Tikhonov and Arsenin [13]). However, under a priori bounds of u, we can restore the stability and, for stable numerical reconstructions of solutions, we can use regularization techniques. There are a large number of works devoting to stable numerical methods. We cannot list all works completely and the following is a partial list: Cheng, Hon, Wei and Yamamoto [2], Hào and Lesnic [5], Klibanov and Santosa [8], Lattes and Lions [9], Reinhardt, Han and Hào [10].

4.1 Conditional stability

First, we mention the conditional stability estimates for the Cauchy problem (3).

Theorem 6 (boundary conditional stability,[12]). *Let* $\eta > \frac{n+2}{2}$. *For* $0 < \kappa_0 < 1$, *there exists a constant* $C > 0$ *such that*

$$\|u\|_{L^\infty(\partial\Omega\setminus\Gamma)} \le C\|u\|_{H^\eta(\Omega)}\left(\log\frac{1}{\|g_1\|_{L^2(\Gamma)}+\|g_2\|_{L^2(\Gamma)}+\|h\|_{L^2(\Omega)}}+\log\frac{1}{\|u\|_{H^\eta(\Omega)}}\right)^{-\kappa_0}.$$

The theorem says that if the norm $\|g_1\|_{L^2(\Gamma)}+\|g_2\|_{L^2(\Gamma)}+\|h\|_{L^2(\Omega)}$ of data tends to zero, then $\|u\|_{L^\infty(\partial\Omega\setminus\Gamma)}$ approaches 0 provided that we know an *a priori* bound for $\|u\|_{H^\eta(\Omega)}$. The rate of convergence of $\|u\|_{L^\infty(\partial\Omega\setminus\Gamma)}$ is logarithmic.

4.2 Reconstruction method

We assume that the problem (3) admits a unique solution $u_0 \in H^{\frac{3}{2}}(\Omega)$ for g_1 and g_2. In this section, we show a reconstruction method by means of the discretized Tikhonov regularization proposed in the previous section. We assume that $\Omega \subset \mathbb{R}^2$ for simplicity. We also assume that there exists a C^∞ map $\Pi\colon [0,1] \to \partial\Omega\setminus\Gamma$ such that Π is injective and $\Pi([0,1]) = \partial\Omega\setminus\Gamma$. Set $\Sigma := \partial\Omega\setminus\Gamma$. Let $\Phi(x,y)\colon [0,1]\times[0,1] \to \mathbb{R}$ be a positive definite kernel on $[0,1]$. Let $\mathcal{H}$ be the RKHS on $[0,1]$ generated by the kernel Φ. We denote $\varphi(\Pi^{-1}(x))$ by $\Pi_*\varphi(x)$ for $\varphi \in \mathcal{H}$ and $x \in \Sigma$. For $m \in \mathrm{N}$, we define a set of points $X_m \subset [0,1]$. We define the finite subspace V_m by $V_m := \mathcal{V}_{X_m}$ and P_m by $P_m := P_{V_m}$, respectively.

We pose the following two assumptions on the positive definite kernel that is satisfied by many type of positive definite kernels [14].

Assumption 7. *We assume that the kernel Φ is uniformly continuous on I.*

Assumption 8. *Suppose there exists a function $p: \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{r\to 0} p(r) = 0$ such that the estimate holds $\|f - P_m f\|_{L^\infty(0,1)} \le p(h_{X_m})\|f\|_{\mathcal{H}}$. for all $f \in \mathcal{H}$. Here $h_{X_m} := \sup_{x\in[0,1]} \min_{x_k\in X_m} |x - x_k|$.*

Firstly, we construct an approximation to $\partial_A u_0|_\Sigma$ of the solution of (3). After obtaining the approximation, we solve a boundary value problem which is well-posed and obtain an approximation to the solution of (3). Thus it suffices to approximate $\partial_A u_0|_\Sigma$.

We define a Hilbert space on Σ by $\mathcal{H}_\Sigma := \{\Pi_*\varphi: \Sigma \to \mathbb{R} \mid \varphi \in \mathcal{H}\}$, equipped with an inner product $(\Pi_*\varphi_1, \Pi_*\varphi_2)_{\mathcal{H}_\Sigma} := (\varphi_1, \varphi_2)_{\mathcal{H}}$, where $\varphi_i \in \mathcal{H}$. It is easy to check that $\mathcal{H}_\Sigma$ is a RKHS generated by the kernel $\Psi(x,y) := \Phi(\Pi^{-1}(x), \Pi^{-1}(y))$.

Let Γ_0 be a relatively open subset of Γ. Let u_0 denote the unique solution of (3). We assume that $\partial_A u_0(\Pi(t)) \in \mathcal{H}$. Suppose that the noisy data g_1^δ and g_2^δ satisfy

$$\|g_1 - g_1^\delta\|_{L^2(\Gamma)} \le \delta, \text{ and } \|g_2 - g_2^\delta\|_{L^2(\Gamma)} \le \delta.$$

We first consider the direct problem

$$\begin{cases} Au = h, & x \in \Omega, \\ \partial_A u|_\Sigma = \theta_1, & \\ u|_{\Gamma_0} = \theta_2, & \\ \partial_A u|_{\Gamma\setminus\Gamma_0} = \theta_3, & \end{cases} \tag{4}$$

for $\theta_1 \in L^2(\Sigma)$, $\theta_2 \in L^2(\Gamma_0)$ and $\theta_1 \in L^2(\Gamma\setminus\Gamma_0)$. We denote the solution of (4) by $u(\theta_1, \theta_2, \theta_3, h)$.

Let L and g^δ be defined, respectively, by

$$L\varphi := u(\varphi, 0, 0, 0)|_{\Gamma\setminus\Gamma_0}, \quad g_\delta = g_1^\delta - u(0, g_1^\delta, g_2^\delta, h)|_{\Gamma\setminus\Gamma_0}.$$

Note that the map $\varphi \in L^2(\Sigma) \to u(\varphi, 0, 0, 0)|_{\Gamma\setminus\Gamma_0} \in L^2(\Gamma\setminus\Gamma_0)$ is compact and injective. In fact, the injectivity follows from the unique continuation (e.g., Isakov [6]). The compactness is seen as follows; the map $\varphi \longrightarrow u(\varphi, 0, 0, 0)$ is continuous from $L^2(\Sigma)$ to $H^1(\Omega)$ by a variational formulation or the Lax-Milgram theorem. Since the embedding $H^{\frac{1}{2}}(\Gamma\setminus\Gamma_0) \longrightarrow L^2(\Gamma\setminus\Gamma_0)$ is compact, we see from the trace theorem that the map is compact. Moreover, the RKHS $\mathcal{H}_\Sigma$ is continuously embedded into $L^2(\Sigma)$. Therefore, L is a linear and injective compact operator from $\mathcal{H}_\Sigma$ to $L^2(\Gamma\setminus\Gamma_0)$. Let K be defined by $K\varphi := L(\Pi_*\varphi)$. It is clear that K is a linear and injective compact operator from $\mathcal{H}$ to $L^2(\Gamma\setminus\Gamma_0)$. Also, we have $g_\delta \in L^2(\Gamma\setminus\Gamma_0)$. We set $g_0 = g_1 - u(0, g_1, g_2, h)|_{\Gamma\setminus\Gamma_0}$.

Lemma 9 ([12]). *Let $\varphi \in \mathcal{H}$. Then $K(\varphi) = g_0$ and $\Pi_*\varphi = \partial_A u_0|_\Sigma$ are equivalent.*

From Lemma 9, the problem of finding $\partial_A u_0|_\Sigma$ from g_1^δ and g_2^δ is equivalent to the problem of finding the solution $\varphi \in \mathcal{H}$ in $K\varphi = g_0$ from g_δ. We solve the problem by the method introduced in section 1; that is, we expand the data g_0^δ in terms of $\{K(\Phi(\cdot, x_k)); x_k \in X_m\}$ on $L^2(\Gamma\backslash\Gamma_0)$. In order to circumvent the instability of the inverse problem, the Tikhonov regularization is applied

$$\min_{\varphi\in V_m} \|K(\varphi) - g_\delta\|^2_{L^2(\Gamma\backslash\Gamma_0)} + \alpha\|\varphi\|^2_{\mathcal{H}_\Sigma},$$

where $\alpha > 0$ is a regularization parameter. We know that there exists a unique minimizer which we denote by $\varphi_{\alpha,m,\delta}$. By $\varphi_{\alpha,m}$, we denote the minimizer when $g_\delta = g_0$.

We can apply Theorem 5 in section 3, we show the convergence of $\varphi_{\alpha,m,\delta}$.

Theorem 10 ([12]). *Under the above settings, we have:*

(i) Let $\lim_{m\to\infty} \alpha_m = 0$. If $p(h_{X_m}) = O(\sqrt{\alpha_m})$. Then, we have

$$\lim_{m\to\infty} \|\Pi_*\varphi_{\alpha,m} - \partial_A u_0\|_{L^2(\Sigma)} = 0.$$

(ii) Let $\lim_{\delta\to 0} m(\delta) = \infty$ and $\lim_{\delta\to 0} \alpha(\delta) = 0$. If $p(h_{X_m}) = O(\sqrt{\alpha})$ and $\delta = O(\sqrt{\alpha})$. Then, we have $\lim_{\delta\to 0} \|\Pi_*\varphi_{\alpha,m,\delta} - \partial_A u_0\|_{L^2(\Sigma)} = 0$.

We solve the boundary value problem

$$\begin{cases} Au = h, & x \in \Omega, \\ \partial_A u|_\Sigma = \Pi_*\varphi_{\alpha,m,\delta}, & \\ u|_{\Gamma_0} = g_1^\delta, & \\ \partial_A u|_{\Gamma\backslash\Gamma_0} = g_2^\delta, & \end{cases} \tag{5}$$

We denote a unique solution of (5) by $u_{\alpha,m,\delta}$. By $u_{\alpha,m}$, we denote the solution obtained by using $\varphi_{\alpha,m}$ and the noise-free data g_1 and g_2 in (5).

The function $u_0 - u_{\alpha,m,\delta}$ satisfies (4) with $\theta_1 = \partial_A u_0 - \Pi_*\varphi_{\alpha,m,\delta}$, $\theta_2 = g_1 - g_1^\delta$ and $\theta_3 = g_2 - g_2^\delta$. Hence, by Theorem 10, we have $\lim_{\delta\to 0} \|u_0 - u_{\alpha,m,\delta}\|_{L^2(\Omega)} = 0$. For given data g_0^δ, g_1^δ and a finite set of points X_m of $[0,1]$, the minimizer $\varphi_{\alpha,m,\delta} \in V_m$ can be written in the form: $\varphi_{\alpha,m,\delta} = \sum_{k=1}^m \lambda_k \Phi(\cdot, x_k)$. The coefficients $\{\lambda_k\}_{k=1}^m$ are obtained by solving the linear system $\dfrac{\partial J(\lambda)}{\partial \lambda_k} = 0$, $k = 1, \dots, m$, where $J(\lambda) := \|K(\sum_{k=1}^m \lambda_k \Phi(\cdot, x_k)) -$

$g_\delta\|^2_{L^2(\Gamma\setminus\Gamma_0)} + \alpha\|\sum_{k=1}^{m}\lambda_k\Phi(\cdot,x_k)\|^2_{\mathcal{H}}$. It is easy to check that the resultant system is

$$(A+\alpha B)\lambda = G_\delta. \tag{6}$$

In (6),

$$\begin{aligned}
[A]_{i,j} &= \int_{\Gamma\setminus\Gamma_0} K(\Phi(\cdot,x_i))K(\Phi(\cdot,x_j))dS, \quad [B]_{i,j} = \Phi(x_i,x_j),\\
[G_\delta]_i &= \int_{\Gamma\setminus\Gamma_0} K(\Phi(\cdot,x_i))g_\delta dS.
\end{aligned}$$

We note that $K(\Phi(\cdot,x_i)) = L(\Pi_*\Phi(\cdot,x_i)),\quad 1\le i\le m$ is the trace on $\Gamma\setminus\Gamma_0$ of the solution u_i of the following direct problem

$$\begin{cases}
Au_i = 0 & \text{in } \Omega,\\
\partial_A u_i|_\Sigma = \Phi(\Pi^{-1}(\cdot),x_i), & \\
u_i|_{\Gamma_0} = 0, & \\
\partial_A u_i|_{\Gamma\setminus\Gamma_0} = 0. &
\end{cases} \tag{7}$$

The direct problem can be solved numerically by using a conventional method such as a finite element method, a finite difference method, a boundary element method, the method of fundamental solution and the Kansa's method, [7], etc.

5 Numerical experiments

In this section, we verify the numerical efficiency of the proposed method for the Cauchy problem (3). We reconstruct an approximate solution to (3) for any given m in X_m. We only focus on the case when $A=\triangle$ and $h=0$, i.e, the Laplace equation. Firstly, we give an approximation to $\partial_A u_0|_\Sigma$. Then, by using such approximation, we solve equation (5) to obtain an approximate solution to (3). The regularization parameter α is chosen by the L-curve method (e.g., [3]).

We consider a two-dimensional case where $\Omega = [-1,1]\times[0,1]$ and two cases of Γ: (i) $\partial\Omega\setminus\Gamma = [-1,1]\times\{1\}$ and (ii) $\Gamma = [-1,1]\times\{0\}$.

We fix the boundary $\Gamma_0 = [-0.1,0.1]\times\{0\}$ in all the cases.

We choose the following functions as test examples:

Example 1 $u_0(x,y) = x^3 - 3xy^2 + e^{2y}\sin 2x - e^y\cos x$.

Example 2 $u_0(x,y) = \cos\pi x\cosh\pi y$.

We use two positive definite kernels among Φ_1 and Φ_2:

Kernel 1 $\Phi_1(t,s) := \exp(-10|t-s|^2)$.

Kernel 2 $\Phi_2(t,s) := \varphi(|t-s|)$, where $\varphi(r) := (1-r)_+^3(3r+1)$ and $t_+ = \max\{t, 0\}$.

Each kernel satisfies the Assumption 8 with $p(r) = C_1 \exp(-\frac{C_2}{r})$ for the Kernel 1 and $p(r) = C_3 r^3$ for the Kernel 2, respectively, where C_1, C_2 and C_3 are positive constants [14, Section 11.4].

For the case (i) $\Gamma = [-1,1] \times \{0\}$, the boundary $\Sigma = \partial\Omega \backslash \Gamma$ is composed by three segments: $\Sigma_1 := \{(s,1); s \in [-1,1]\}$, $\Sigma_2 := \{(-1,s); s \in [0,1]\}$ and $\Sigma_3 := \{(1,s); s \in [0,1]\}$. We define maps $\Pi_i \colon [0,1] \to \Sigma_i$, $i = 1,2,3$ by $\Pi_1(t) = (-1,t)$, $\Pi_2(t) = (-1+2t, 1)$ and $\Pi_3(t) = (1,t)$ for $t \in [0,1]$.

We take two finite sets of points X_{10} and X_{20} in $[0,1]$. The fill distances of both $\Pi_1(X_{10})$ and $\Pi_3(X_{10})$ are equal to that of $\Pi_2(X_{20})$.

The noisy data $\{g_1^\delta, g_2^\delta\}$ are obtained by adding random numbers to the exact data $\{g_1, g_2\} = \{u_0|_\Gamma, \partial_A u_0|_\Gamma\}$ by

$$g_i^\delta(\xi) = g_i(\xi) + \frac{\delta}{100} \max_{z\in\Gamma} |g_i(z)| \mathrm{rand}(\xi), \quad i = 1,2,$$

for $\xi \in \Gamma$, where $\mathrm{rand}(\xi)$ is a random number between $[-1,1]$ and $\delta\% \in \{0,1,5,10\}$ is the noise level.

For all given noisy data $\{g_1^\delta, g_2^\delta\}$ with various noisy levels, we apply Algorithm to obtain an approximate solution to u_0 in each example. We denote by u_{Φ_i} the approximate solution obtained with using the kernel Φ_i, $i = 1,2$ in Algorithm. For the numerical error estimations, we compute the relative error by of u_{Φ_i} over the whole domain Ω:

$$E_r(u_{\Phi_i}) := \frac{\|u_0 - u_{\Phi_i}\|_{L^2(\Omega)}}{\|u_0\|_{L^2(\Omega)}},$$

for $i = 1,2$. Table 1 shows the relative errors for Example 1 and Example 2. In Figure 1, we show the solution u_0 in Example 2 for the comparison to approximate solution u_{Φ_2}. The solutions u_{Φ_2} obtained by using different noisy data with noise level $\delta = 0,1,5,10$ are given in Figure 2 - Figure 5, respectively.

In order to study the error profiles of our numerical solution to u_{Φ_2}, in Figure 6 and Figure 7, we draw the absolute error

$$E_a(x,y) := |u_0(x,y) - u_{\Phi_2}(x,y)|, \quad (x,y) \in \Omega.$$

In this experiment, the noise level is set to be $\delta = 10$ and both Example 1 and Example 2 are tested. We observe that the errors becomes larger near the boundary Σ in the both examples. This corresponds to the conditional stability estimate up to the boundary as we stated in Theorem 6 where the rate of the convergence to the exact solution is only logarithmic. By the interior conditional stability estimate for Cauchy problem [6], we may expect that the accuracy of the numerical solution will be improved in a small part of the subset $\omega \subset \Omega$ whose boundary $\partial\omega$ does not touch Σ. In [8], the reconstruction was done in a subdomain ω for the same Cauchy problem for the Laplace equation. For comparisons, we choose the same subdomain ω:

$$\omega := \{(x,y); y + 0.6\left(\frac{x}{0.6}\right)^2 - 0.6 \leq 0,\ y \geq 0\}$$

and consider the relative error in ω

$$e_r(u_{\Phi_i}) := \frac{\|u_0 - u_{\Phi_i}\|_{L^2(\omega)}}{\|u_0\|_{L^2(\omega)}}, \quad i = 1,2.$$

In Table 2, we can see that all the accuracies have improved.

Finally, we compute the numerical approximate solution to u_0 when the Cauchy data is given on the boundary $\Sigma = \{(x,1); x \in [-1,1]\}$. Table 3 and Table 4 show the relative errors in each domain respectively.

	Example1		Example2	
Noise	$E_r(u_{\Phi_1})$	$E_r(u_{\Phi_2})$	$E_r(u_{\Phi_1})$	$E_r(u_{\Phi_2})$
0%	0.0428	0.0338	0.0919	0.0667
1%	0.0507	0.0606	0.1099	0.0781
5%	0.2449	0.2340	0.3055	0.3186
10%	0.2797	0.2682	0.3410	0.3149

Table 1: The relative errors u_{Φ_i} $i = 1,2$ on the whole domain Ω when the Cauchy data are given on the boundary $\Gamma = [-1,1] \times \{0\}$.

	Example1		Example2	
Noise	$e_r(u_{\Phi_1})$	$e_r(u_{\Phi_2})$	$e_r(u_{\Phi_1})$	$e_r(u_{\Phi_2})$
0%	0.0044	0.0040	0.0023	0.0019
1%	0.0041	0.0074	0.0072	0.0052
5%	0.0717	0.0677	0.0638	0.0786
10%	0.0879	0.0830	0.0768	0.0763

Table 2: The relative errors u_{Φ_i}, $i = 1, 2$, in the interior part ω where the Cauchy data is given on the boundary $\Gamma = [-1, 1] \times \{0\}$.

	Example1		Example2	
Noise	$E_r(u_{\Phi_1})$	$E_r(u_{\Phi_2})$	$E_r(u_{\Phi_1})$	$E_r(u_{\Phi_2})$
0%	0.0069	0.0043	0.0037	0.0044
1%	0.0153	0.0106	0.0166	0.0046
5%	0.0375	0.0218	0.0361	0.0198
10%	0.0414	0.0425	0.0539	0.0292

Table 3: The relative errors u_{Φ_i}, $i = 1, 2$, on the whole domain Ω where the Cauchy data is given on the boundary Γ such that $\partial\Omega\backslash\Gamma = [-1, 1] \times \{1\}$.

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	Example1		Example2	
Noise	$e_r(u_{\Phi_1})$	$e_r(u_{\Phi_2})$	$e_r(u_{\Phi_1})$	$e_r(u_{\Phi_2})$
0%	0.0012	0.0010	0.0034	0.0037
1%	0.0078	0.0054	0.0078	0.0046
5%	0.0176	0.0098	0.0276	0.0115
10%	0.0207	0.0200	0.0406	0.0138

Table 4: The relative errors u_{Φ_i}, $i = 1, 2$, in the interior part ω where the Cauchy data is given on the boundary Γ such that $\partial\Omega \backslash \Gamma = [-1, 1] \times \{1\}$.

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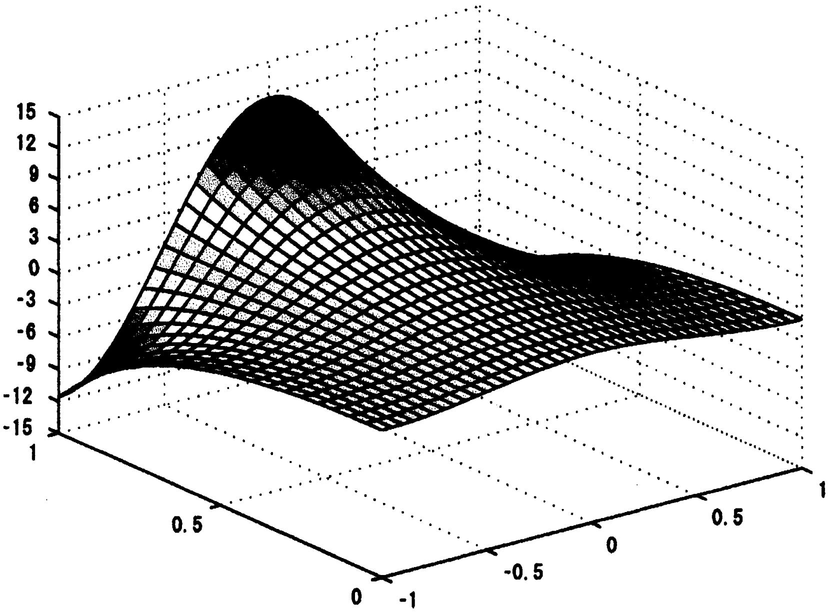

Figure 1: Surface plot for the function $u_0(x, y) = \cos \pi x \cosh \pi y$ in example 2.

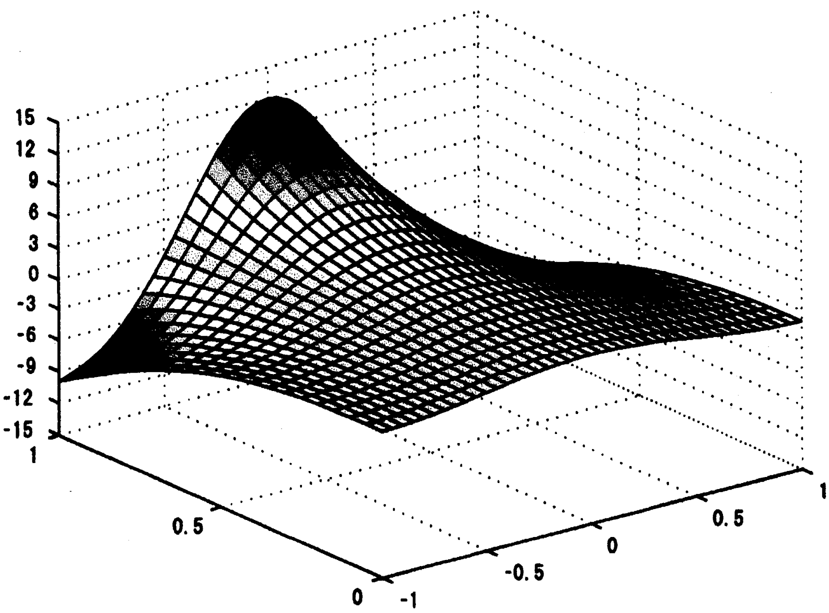

Figure 2: Numerical approximate solution u_{Φ_2} to the solution of example 2 using noisy data when $\delta = 0$

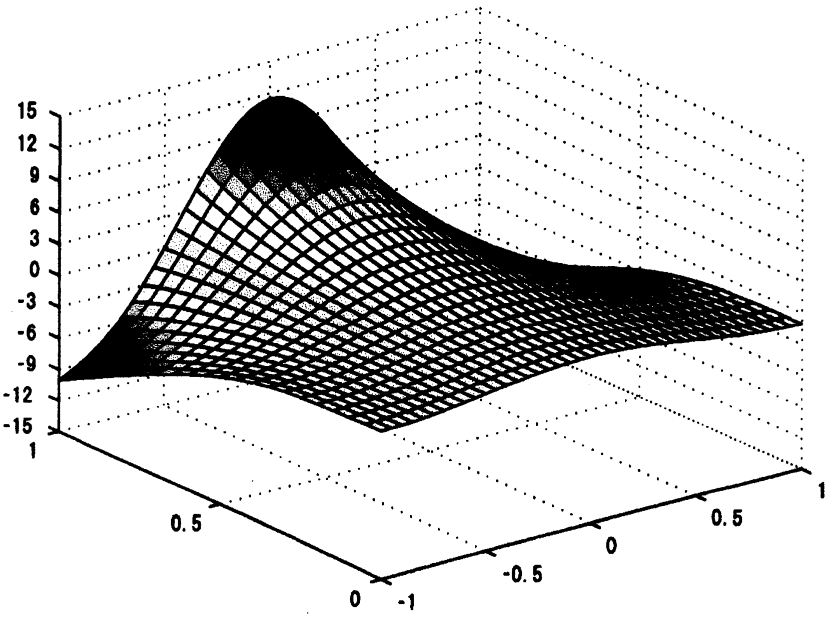

Figure 3: Numerical approximate solution u_{Φ_2} to the solution of example 2 using noisy data when $\delta = 1$

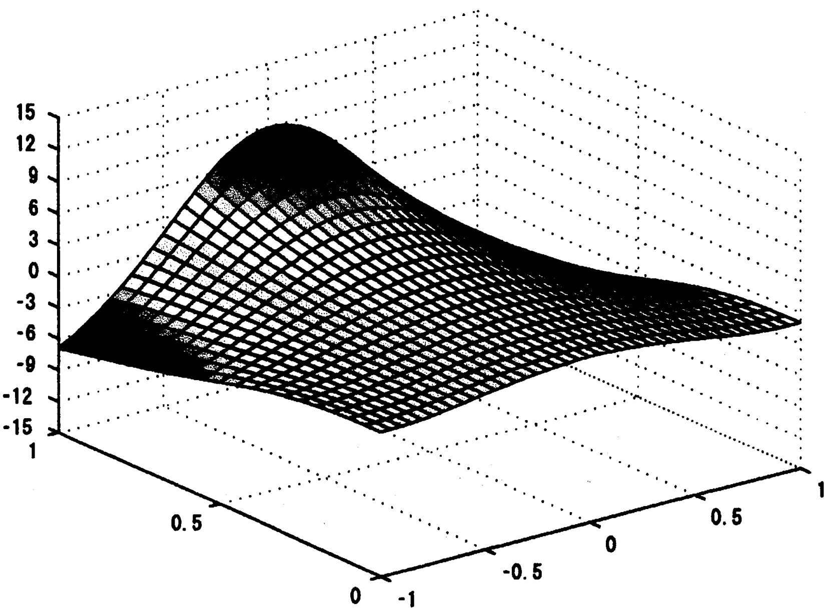

Figure 4: Numerical approximate solution u_{Φ_2} to the solution of example 2 using noisy data when $\delta = 5$

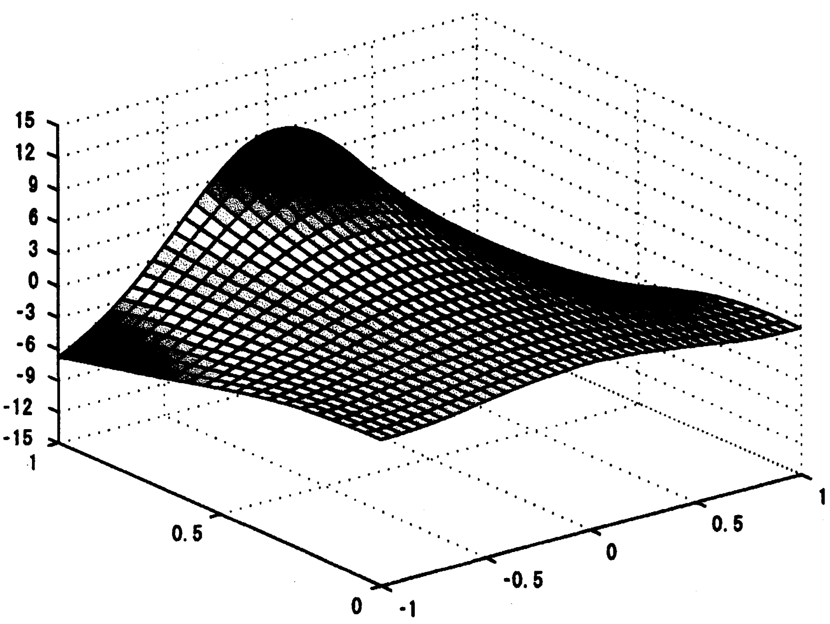

Figure 5: Numerical approximate solution u_{Φ_2} to the solution of example 2 using noisy data when $\delta = 10$

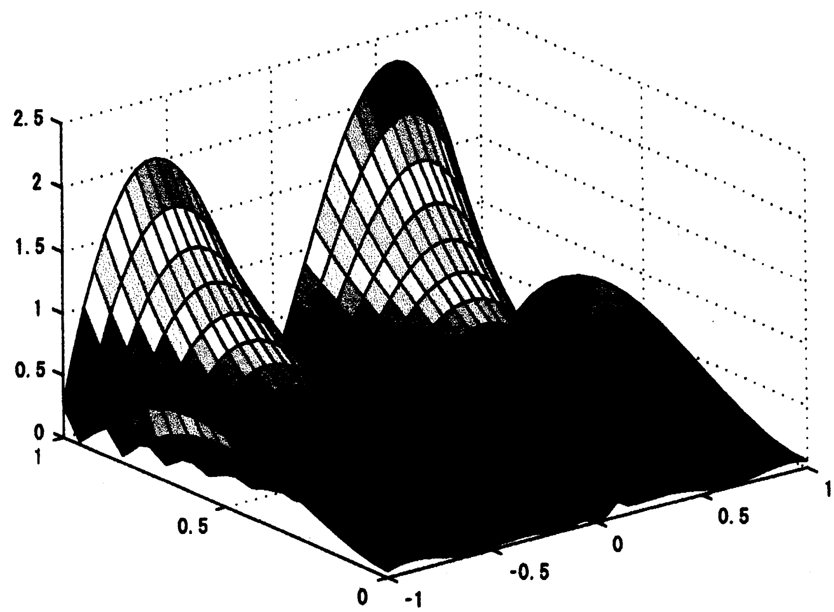

Figure 6: Absolute error $|u_0(x,y) - u_{\Phi_2}(x,y)|$ by the Cauchy data on $\Gamma = [-1,1] \times \{0\}$ with 10% noise.

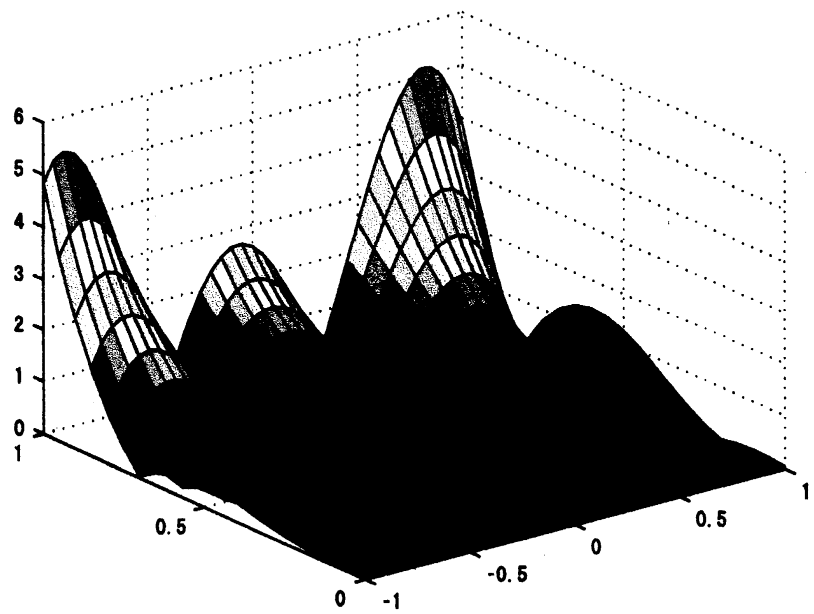

Figure 7: Absolute error $|u_0(x,y) - u_{\Phi_2}(x,y)|$ by the Cauchy data on $\Gamma = [-1,1] \times \{0\}$ with 10% noise.